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GAME PROBLEMS ON ENCOUNTER WITH m TARGET SETS

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An encounter — evasion differential game for several target sets is analyzed. The players' piecewise-postition strategies are determined and it is established that an ε-equilibrium situation exists in the class of these strategies. The material in this paper is closely related with the investigations in [1, 2].

1. Let the motion of a controlled system be described by the equation

$$x^{\cdot} = f(t, x, u, v), \quad f: [t_0, \infty) \times R^n \times P \times Q \to R^n \tag{1.1}$$

where f is a continuous function and $P \subset R^p$ and $Q \subset R^q$ are compacta. It is assumed that

$$|x' f(t, x, u, v)| \leqslant \varkappa (1 + \|x\|^2), \quad (t, x, u, v) \in [t_0, \infty) \times R^n \times P \times Q$$

$$\| f(t, x^{(1)}, u, v) - f(t, x^{(2)}, u, v)\| \leqslant \lambda_G \| x^{(1)} - x^{(2)} \|$$

$$(t, x^{(i)}, u, v) \in G \times P \times Q, \quad i = 1, 2$$

where $x'f$ is the scalar product of vectors x and f, $\|x\|^2 = x'x$, $\varkappa$ is a constant number and G is any bounded domain from $[t_0, \infty) \times R^n$. It is assumed as well that the condition

$$\min_{u \in P} \max_{v \in Q} s' f(t, x, u, v) = \max_{v \in Q} \min_{u \in P} s' f(t, x, u, v) \tag{1.2}$$

$$s \in R^n, \quad (t, x) \in [t_0, \infty) \times R^n$$

is fulfilled.

Compacta M_k and N, $k = 1, \ldots, m$ (M_k are the target sets and N is a phase limitation), are specified in space R^{n+1}. For a continuous function $x[\cdot]$: $[t_0, \infty) \to R^n$ we define the set

$$T_k(x[\cdot]) = \{\tau: (\tau, x[\tau]) \in M_k, \ (t, x[t]) \in N, \ t_0 \leqslant t \leqslant \tau\}$$

We further set

$$\tau_k(x[\cdot]) = \begin{cases} \min T_k(x[\cdot]), & T_k(x[\cdot]) \neq \varnothing \\ \infty, & T_k(x[\cdot]) = \varnothing \end{cases}$$

Here the symbol $\min T$ denotes the smallest of the numbers occurring in set T. Thus, $\tau_k(x[\cdot])$ is the instant that point $(t, x[t])$ first hits the set M_k under the condition that the inclusion $(t, x[t]) \in N$ was fulfilled up to contact with M_k.

The payoff γ in the differential game being examined is determined by the equality

$$\gamma(x[\cdot]) = \sigma(\tau_1(x[\cdot]), \ldots, \tau_m(x[\cdot])) \tag{1.3}$$
$$(x[\cdot] : [t_0, \infty) \to R^n, \ \sigma : [t_0, \infty]^m \to (-\infty, \infty])$$

Here $x[\cdot]$ is a realized motion of the system and σ is a prescribed function satisfying the following conditions:

1) function σ takes finite values and is continuous on the set $[t_0, \infty)^m$;
2) $\sigma(\tau_1, \ldots, \tau_m) = \infty$ if even one $\tau_k = \infty$;
3) the set $\sigma^{-1}((-\infty, c])$ is bounded for any finite number c;
4) the inequality
$\sigma(\tau_1, \ldots, \tau_{i-1}, \tau_i', \tau_{i+1}, \ldots, \tau_m) \leqslant \sigma(\tau_1, \ldots, \tau_{i-1}, \tau_i'', \tau_{i+1}, \ldots, \tau_m)$
is valid for any collections $(\tau_1, \ldots, \tau_{i-1}, \tau_i', \tau_{i+1}, \ldots, \tau_m)$ and $(\tau_1, \ldots, \tau_{i-1}, \tau_i'', \tau_{i+1}, \ldots, \tau_m)$, where $\tau_i' \leqslant \tau_i''$.

The conditions indicated here are satisfied, for instance, by the function $\sigma(\tau_1, \ldots, \tau_m) = \max \tau_k$ for $k = 1, \ldots, m$. In this case $\gamma(x[\cdot]) - t_0$ is the time by which the motion $x[\cdot]$ makes contact with all sets M_k $(k = 1, \ldots, m)$ inside N. It is assumed that the first player, governing the control u, strives to minimize the value of payoff γ, while the second player, choosing the control v, maximizes the value of γ.

The functional γ of (1.3) is lower-semicontinuous; therefore (see [1]), in the game being analyzed an ε-equilibrium situation exists in the class of pure strategies $U \div u(x[\cdot; t_0, t])$ and $V \div v(x[\cdot; t_0, t])$ with complete memory. It is established below that the ε-equilibrium situation is preserved if the players use not all the information on the trajectory $x[\cdot; t_0, t] = (x[\xi], t_0 \leqslant \xi \leqslant t)$ realized by instant t, but only the information on the position $(t, x[t])$ realized and on certain numbers t_k defined for this trajectory. Informally these numbers can be defined as the instants of encounter of position $(t, x[t])$ with the target sets M_k. Pure position strategies $U \div u(t, x)$ and $V \div v(t, x)$ are used in each interval between such instants. Thus, an equilibrium situation in the game being examined is achieved in the class of piecewise-position strategies.

2. Let us present the formal definitions of the piecewise-position strategies and of the motions generated by them. The collection of mappings

$$\alpha : (t, x, t_1, \ldots, t_m) \to \alpha(t, x, t_1, \ldots, t_m) \tag{2.1}$$
$$\varphi_k : x[\cdot; t_0, t] \to \varphi_k(x[\cdot; t_0, t]) \quad (k = 1, \ldots, m)$$
$$t \in [t_0, \infty), \quad x[\cdot; t_0, t] \in C^n[t_0, t]$$

is called the first player's piecewise-position strategy U. Here $C^n[t_0, t]$ is the space of continuous functions $x[\cdot; t_0, t] : [t_0, t] \to R^n$; the functionals φ_k are defined on the set $C_* = \{\bigcup C^n[t_0, t] : t_0 \leqslant t < \infty\}$ and take values from $[t_0, \infty]$; the function α is defined on the set $[t_0, \infty) \times R^n \times [t_0, \infty]^m$ and takes values from compactum P. Each of the functionals φ_k satisfies the following condition. Let $t^* \in [t_0, \infty)$, $x^*[\cdot; t_0, t^*] \in C^n[t_0, t^*]$, $t \in [t_0, t^*]$ and $x^*[\cdot; t_0, t]$ be the restriction of function $x^*[\cdot; t_0, t^*]$ on interval $[t_0, t]$. Then either $\varphi_k(x^*[\cdot; t_0, t^*]) = \infty$, and in this case $\varphi_k(x^*[\cdot; t_0, t]) = \infty$ for all $t \in [t_0, t^*]$, or $\varphi_k(x^*[\cdot; t_0, t^*]) = t_k^* \leqslant t^*$, and in this case $\varphi_k(x^*[\cdot; t_0, t]) = \{\infty$ when $t_0 \leqslant$

$t < t_k^*$, t_k^* when $t_k^* \leqslant t \leqslant t^*\}$. Thus, functional φ_k takes no more than two values along any motion $x[\cdot]$, and the change from one value to the other can take place no more than once. The second player's piecewise-position strategy V is defined analogously. The mappings

$$\beta : [t_0, \infty) \times R^n \times [t_0, \infty]^m \to Q \qquad (2.2)$$
$$\psi_k : C_* \to [t_0, \infty] \quad (k = 1, \ldots, m)$$

defining V satisfy the same conditions (2.1) indicated for U.

The motions generated by U of (2.1) are introduced in the following manner. Suppose that the first player has chosen a partitioning $\Delta = \{[\tau_i, \tau_{i+1}): i = 0, 1, \ldots; \tau_i \to \infty$ as $i \to \infty$; $\tau_0 = t_0\}$. We assume that under this partitioning the U of (2.1) forms a piecewise-constant control $u_\Delta[t]$ $(t \geqslant t_0)$ by the rule

$$u_\Delta[t] = \alpha(\tau_i, x_\Delta[\tau_i; t_0, \tau_i], \varphi_1(x_\Delta[\cdot; t_0, \tau_i]), \ldots$$
$$\varphi_m(x_\Delta[\cdot; t_0, \tau_i])) \quad \tau_i \leqslant t < \tau_{i+1} \quad (i = 0, 1, \ldots)$$

where $x_\Delta[\cdot; t_0, \tau_i]$ is a solution of system (1.1), which was realized on tne interval $[t_0, \tau_i]$ and corresponding to control $u_\Delta[t]$ and to some measurable control $v[t] \in Q$ $(t_0 \leqslant t < \tau_i)$ selected by the second player. The motions $x_\Delta[t]$ $(t \geqslant t_0)$ thus defined are called approximate and are denoted by the symbol $x_\Delta[\cdot; t_0, x_0, U, v[\cdot]]$, where $x_0 = x_\Delta[t_0]$ is the initial state and $v[\cdot]$ is a realization of the second player's control.

By the symbol $X(t_0, x_0, U)$ we denote the collection of functions $x[\cdot]:$ $[t_0, \infty) \to R^n$ for each of which there exists a sequence of approximate motions $x_{\Delta_j}[\cdot; t_0, x_{0,j}, U, v_j[\cdot]]$, converging uniformly on every finite interval $[t_0, t_*]$ to function $x[\cdot]$ and satisfying the conditions $x_{0,j} \to x_0$ and $\sup_i (\tau_{i+1,j} - \tau_{i,j}) \to 0$ as $j \to \infty$. The elements of set $X(t_0, x_0, U)$ are called the system's motions generated by the first player's piecewise-position strategy U. The motions $x[\cdot] \in X(t_0, x_0, V)$ generated by the second player's piecewise-position strategy V are introduced analogously. We note that any pair U and V can be realized simultaneously in the differential game, since the motions $x[\cdot] \in X(t_0, x_0, U) \cap X(t_0, x_0, V)$ generated by such a pair (U, V) can always be defined.

T h e o r e m. Let condition (1.2) be fulfilled. Then an ε-equilibrium situation exists in the class of piecewise-position strategies U and V of forms (2.1) and (2.2), i.e., the first player's piecewise-position strategy U° exists and for any $\varepsilon > 0$ the second player's piecewise-position strategy V^ε exists, such that

$$\sup \gamma(X(t_0, x_0, U^\circ)) = \min_U \sup \gamma(X(t_0, x_0, U)) = \gamma_0$$
$$\inf \gamma(X(t_0, x_0, V^\varepsilon)) + \varepsilon \geqslant \sup_V \inf \gamma(X(t_0, x_0, V)) = \gamma_0$$

This theorem can be proved by the scheme in [1]. Strategies U° and V^ε can be determined as strategies extremal to appropriate bridges. Let us describe the extremal strategy U°. In this strategy the functionals φ_k° associate with function $x[\cdot; t_0, t]$ either a number t_k $(t_k \leqslant t)$, which can informally be defined as the instant that the point $(\xi, x[\xi])$ first encountered set M_k, or the improper number ∞, if this encounter did not take place on the interval $[t_0, t]$. The function α° is defined as follows. In the space of positions (t, x) we define a u-stable W_0 as well as the u-stable bridges $W_j(t_{k_1}, \ldots, t_{k_j})$ corresponding to the collections of parameters t_{k_1}

$\leqslant t_{k_2} \leqslant \ldots \leqslant t_{k_j} < \infty$ $(1 \leqslant j \leqslant m-1)$. For the collection $t_k = \infty$, $k = 1, \ldots, m$, the function

$$\alpha^\circ(\cdot, t_1, \ldots, t_m): (t, x) \to \alpha^\circ(t, x, t_1, \ldots, t_m) \qquad (2.3)$$

is defined as the position strategy extremal to bridge W_0. For the collection $(t_1, \ldots, t_m)$, where $t_{k_1} \leqslant t_{k_2} \leqslant \ldots \leqslant t_{k_j} < \infty$, and the remaining $t_k = \infty$, the function $\alpha^\circ(\cdot, t_1, \ldots, t_m)$ of (2.3) is the position strategy extremal to bridge $W_j(t_{k_1}, \ldots, t_{k_j})$. For the collection $(t_1, \ldots, t_m)$, where $t_k < \infty$, $k = 1, \ldots, m$, the function $\alpha^\circ(\cdot, t_1, \ldots, t_m)$ of (2.3) is chosen arbitrarily.

Thus, on the interval $[t_0, t_{k_1})$ the control $u[t]$ is formed as a position strategy extremal to bridge W_0; here t_{k_1} is the instant that an encounter first occurs with one of sets M_k (with set M_{k_1}). Then, on the next interval $[t_{k_1}, t_{k_2})$ before the encounter with the next set (with set M_{k_2}) the control $u[t]$ is called the position strategy extremal to bridge $W_1(t_{k_1})$, and so on. We note that the bridges used here are constructed sequentially, beginning with the determination of bridges $W_{m-1}(t_{k_1}, \ldots, t_{k_{m-1}})$ and terminating on the target set M_{k_m} no later than at the instant t_{k_m}. The parameters $(t_1, \ldots, t_m)$ indicated here are such that $\sigma(t_1, \ldots, t_m) \leqslant c$, where c is some prescribed number, this being the result guaranteed to the first player if he uses the extremal strategy U°. Next, all possible bridges $W_{m-2}(t_{k_1}, \ldots, t_{k_{m-2}}), \ldots, W_1(t_{k_1})$ and W_0 are determined in succession. These bridges are such that the extremal strategy U° leads system (1.1) from each of the bridges W_j onto one of the bridges W_{j+1} and simultaneously onto one of the remaining target sets M_k; by the same token strategy U° ensures the solution of the problem facing the first player.

3. Let us consider the case when the fulfilment of condition (1.2) is not presumed. We define a strategy U^v as the collection of m functionals φ_k of the form indicated above and of function $\alpha^v : [t_0, \infty) \times R^n \times Q \times [t_0, \infty]^m \to P$. We assume that this function is Borel-measurable in the variable $v \in Q$. We note that for fixed values of $t_1, \ldots, t_m$ the function $\alpha^v(\cdot, t_1, \ldots, t_m) : [t_0, \infty) \times R^n \times Q \to P$ is a counter-strategy (see [1], p. 356). To determine the approximate motions $x_\Delta[\cdot; t_0, x_0, U^v, v[\cdot]]$ we assume that for a chosen partitioning $\Delta = \{[\tau_i, \tau_{i+1}): i = 0, 1, \ldots\}$ the strategy U^v forms the first player's control by the rule

$$u_\Delta[t] = \alpha^v(\tau_i, x_\Delta[\tau_i; t_0, \tau_i], v[t], \varphi_1(x_\Delta[\cdot; t_0, \tau_i]), \ldots$$
$$\varphi_m(x_\Delta[\cdot; t_0, \tau_i])), \tau_i \leqslant t < \tau_{i+1} \quad (i = 0, 1, \ldots)$$

where $v[t]$ $(t \geqslant t_0)$ is a measurable realization of the second player's control. Further, just as in the case of piecewise-position strategy U, we determine the set $X(t_0, x_0, U^v)$ of motions generated by strategy U^v. Strategy U^v can be realized in pair with V. The theorem on the existence of the ε-equilibrium situation is valid for the differential game (1.1), (1.3) analyzed in the class of first player's strategies U^v mentioned here and in the class of second player's piecewise-position strategies V. The ε-equilibrium situation obtains as well for the class of first player's strategies U and of second player's strategies V^u. The definition of these strategies V^u is obtained from the definition of V by replacing in (2.2) the function β by the function $\beta^u : [t_0, \infty) \times R^n \times P \times [t_0, \infty]^m \to Q$. The theorem on the existence of the ε-equilibrium situation in differential game (1.1), (1.3) is valid also for the class of mixed piecewise-position strategies $\bar{U}$ and $\bar{V}$ of both players. To determine these strategies the functions α and β in (2. 1) and (2. 2) should be replaced by the functions $\bar{\alpha} : [t_0, \infty) \times R^n \times [t_0, \infty]^m \mapsto \bar{P}$ and $\bar{\beta} : [t_0, \infty) \times R^n \times [t_0, \infty]^m \rightarrow \bar{Q}$, where $\bar{P}$ and $\bar{Q}$ are sets of probability measures normed on compacta P and Q, respectively. In the time intervals wherein not even one of the functionals φ_k (or ψ_k) changes its value, the strategy $\bar{U}$ (or $\bar{V}$)forms the system's motions as a mixed position strategy (see [1]).

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